

# Locally trivial quantum vector bundles and associated vector bundles

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## Abstract

We define locally trivial quantum vector bundles (QVB) and construct such QVB associated to locally trivial quantum principal fibre bundles. The construction is quite analogous to the classical construction of associated bundles. A covering of such bundles is induced from the covering of the subalgebra of coinvariant elements of the principal bundle. There exists a differential structure on the associated vector bundle coming from the differential structure on the principal bundle, which allows to define connections on the associated vector bundle associated to connections on the principal bundle.

This is the third in a series of papers devoted to locally trivial quantum bundles, following [3] and [4]. The main aim of the present paper is to define associated bundles and connections in the scheme of locally trivial quantum principal bundles of [2] and [4]. Again we follow the idea of gluing all objects from locally given objects. We note that associated bundles have been defined in [6], [7] and [8] as associated bimodules of colinear maps (intertwiners) and in [1], [2], [9] as cotensor products. Both definitions are equivalent (by some duality argument). We use the second definition.

We start with some general considerations about covering und gluing of modules. Then we define QVB over algebras with a complete covering. They have as typical fibre a usual vector space, and possess local trivializations with suitable properties. Any LC differential algebra on the basis gives rise to a certain “differential structure” on the QVB, which is an analogue of a module of differential form valued sections in the classical case. Given such a structure, we define the notions of connection (as covariant derivative) and curvature on a QVB. Given a locally trivial quantum principal bundle and a left coaction of its structure group (Hopf algebra) on some vector space, we define an associated QVB as a cotensor product. This is indeed a locally trivial QVB in the sense of our definition, whose transition functions come from the transition functions of the principal bundle in the usual way.

As is known from [4], there is a maximal embeddable LC differential algebra related to the differential structure of a locally trivial quantum principal bundle. The differential structure on the QVB defined by this LC differential algebra is isomorphic to the module of horizontal forms “of type  $\rho$ ” on the principal bundle, where  $\rho$  is the left coaction defining the associated bundle. Finally, we show that to every connection on the principal bundle there can be associated a

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connection on the associated QVB. Locally, the associated connection and its curvature are nicely determined by the local connection and curvature forms of the principal connection.

## 1 Coverings of modules

First we give a definition of modules over an algebra being equivalent to the usual one. For a vector space  $E$  we denote by  $End(E)$  the algebra of linear endomorphisms of  $E$ .

**Definition 1** *Let  $E$  be a vector space, let  $B$  be an algebra and  $\kappa : B \longrightarrow End(E)$  a linear map.  $(E, B, \kappa)$  is called left module if  $\kappa$  satisfies  $\kappa(ab) = \kappa(a)\kappa(b)$ ,  $a, b \in B$ .  $(E, B, \kappa)$  is called right module if  $\kappa$  satisfies  $\kappa(ab) = \kappa(b)\kappa(a)$ . For a linear subspace  $Q \subset E$ ,  $(Q, B, \kappa)$  is a submodule if  $\kappa(B)(Q) \subset Q$ .*

**Definition 2** *Let  $(E, B, \kappa)$  be a left (right) module.  $(E, B, \kappa)$  is called faithful if  $ker \kappa = 0$ .*

In analogy to the case of algebras one can define coverings of modules. They will be needed in the definition of quantum vector bundles.

**Definition 3** *Let  $(E, B, \kappa)$  be a left (right) module and let  $(Q_i)_{i \in I}$  be a finite family of left (right) submodules of  $E$ .  $(Q_i)_{i \in I}$  is called covering of  $E$  if  $\bigcap_i Q_i = 0$ .*

Obviously, as in the case of algebras, for a given covering  $(Q_i)_{i \in I}$  of a module  $(E, B, \kappa)$  one obtains a family of vector spaces  $E_i := E/Q_i$  with corresponding projections  $q_i : E \longrightarrow E_i$ . Since  $Q_i$  are submodules, there exist linear maps  $\kappa_i : B \longrightarrow End(E_i)$  defined by

$$\kappa_i(a) \circ q_i = q_i \circ \kappa(a) \quad a \in B$$

such that  $(E_i, B, \kappa_i)$  are left or right modules respectively.

Remark: Since  $\kappa_i$  are homomorphisms for left modules and antihomomorphism for right modules,  $ker \kappa_i$  are ideals in  $B$ .

**Definition 4** *A covering  $(Q_i)_{i \in I}$  of a left (right) module  $(E, B, \kappa)$  is called nontrivial if  $ker \kappa_i \neq 0 \quad \forall i \in I$ .*

**Proposition 1** *Let  $(E, B, \kappa)$  be a faithful left (right) module and let  $(Q_i)_{i \in I}$  be a covering of  $E$ . Then  $(ker \kappa_i)_{i \in I}$  is a covering of  $B$ . The covering  $(Q_i)_{i \in I}$  is nontrivial if the covering  $(ker \kappa_i)_{i \in I}$  is nontrivial.*

Proof: We have to prove that  $\bigcap_i ker \kappa_i = 0$ . It is easy to verify that

$$ker \kappa_i = \{a \in B \mid \kappa(a)(E) \subset Q_i\}.$$

Now it is clear that

$$\bigcap_i ker \kappa_i = \{a \in B \mid \kappa(a)(E) \subset Q_i \quad \forall i \in I\}$$

and since  $\bigcap_i Q_i = 0$  and  $(E, B, \kappa)$  is faithful it follows  $\bigcap_i ker \kappa_i = 0$ . □

Let  $B_i := B/ker \kappa_i$ . It is obvious that the modules  $(E_i, B_i, \tilde{\kappa}_i)$ , where  $\tilde{\kappa}_i$  is defined by

$$\tilde{\kappa}_i \circ \pi_i = \kappa_i,$$

are faithful. Let

$$\begin{aligned} q_{ij} : E &\longrightarrow E/(Q_i + Q_j) := E_{ij} \\ q_j^i : E_i &\longrightarrow E_{ij} \end{aligned}$$

be the canonical projections. Assume that  $(E, B, \kappa)$  is a faithful left (right) module and  $(Q_i)_{i \in I}$  is a covering of  $E$ . One has the vector space

$$E_c := \{(e_i)_{i \in I} \in \bigoplus_{i \in I} E_i \mid q_j^i(e_i) = q_i^j(e_j)\} \quad (1)$$

and an injective homomorphism

$$\mathcal{K} : E \longrightarrow E_c$$

by  $\mathcal{K}(e) = (q_i(e))_{i \in I}$ .

**Proposition 2** *Let  $(E, B, \kappa)$  be a faithful left (right) module and let  $(Q_i)_{i \in I}$  be a covering of  $E$ . Let  $B_c$  be the covering completion of  $B$  with respect to  $(\ker \kappa_i)_{i \in I}$ . Then there exists a linear map  $\kappa_c : B_c \longrightarrow \text{End}(E_c)$  such that  $(E_c, B_c, \kappa_c)$  is a faithful left (right) module satisfying*

$$\mathcal{K}(\kappa(a)(e)) = \kappa_c(K(a))(\mathcal{K}(e)), \quad a \in B, \quad e \in E,$$

where  $K : B \longrightarrow B_c$  is the injective Homomorphism defined by  $K(a) = (\pi_i(a))_{i \in I}$ . Proof: Since  $(Q_i)_{i \in I}$  is a family of submodules, there exist linear maps  $\kappa_{ij} : B \longrightarrow \text{End}(E_{ij})$  defined by

$$\kappa_{ij}(a) \circ q_{ij} = q_{ij} \circ \kappa(a), \quad a \in B$$

such that  $(E_{ij}, B, \kappa_{ij})$  is a left (right) module. Let  $B_{ij} := B/(\ker \kappa_i + \ker \kappa_j)$ . Now one can define the linear map  $\tilde{\kappa}_{ij} : B_{ij} \longrightarrow \text{End}(E_{ij})$  by the formula

$$\tilde{\kappa}_{ij} \circ \pi_{ij} = \kappa_{ij},$$

thus  $(E_{ij}, B_{ij}, \tilde{\kappa}_{ij})$  is a left (right) module. It is easy to verify that  $\tilde{\kappa}_{ij}$  satisfies

$$q_j^i \circ \tilde{\kappa}_i(a) = \tilde{\kappa}_{ij}(\pi_j^i(a)) \circ q_j^i, \quad a \in B_i.$$

Recall that  $B_c := \{(a_i)_{i \in I} \in \bigoplus_{i \in I} B_i \mid \pi_j^i(a_i) = \pi_i^j(a_j)\}$ . We define  $\kappa_c : B_c \longrightarrow \text{End}(E_c)$  as follows:

$$\kappa_c((a_i)_{i \in I})(e_i) = (\tilde{\kappa}_i(a_i)(e_i))_{i \in I}. \quad (2)$$

One has to prove that the image of  $\kappa_c(a_c)$  lies in  $E_c$  for all  $a_c \in B_c$ :

$$\begin{aligned} q_j^i(\tilde{\kappa}_i(a_i)(e_i)) &= \tilde{\kappa}_{ij}(\pi_j^i(a_i))(q_j^i(e_i)) \\ &= \tilde{\kappa}_{ij}(\pi_i^j(a_j))(q_i^j(e_j)) \\ &= q_i^j(\tilde{\kappa}_j(a_j)(e_j)). \end{aligned}$$

The other properties of  $\kappa_c$  follow from the definition. □

**Definition 5** *Let  $(E, B, \kappa)$  be a faithful left (right) module and let  $(Q_i)_{i \in I}$  be a covering of  $E$ . The covering  $(Q_i)_{i \in I}$  is called complete, if the family of ideals  $(\ker \kappa_i)_{i \in I}$  is a complete covering of  $B$  and the injective linear map  $\mathcal{K} : E \longrightarrow E_c$  is a left (right) module isomorphism.*

## 2 Locally trivial quantum vector bundles and associated vector bundles

On the algebraic level, the notion of vector bundle is in the classical case related to the notion of section of a vector bundle. Let  $M$  be a manifold, let  $C(M)$  be the algebra of continuous functions over  $M$  and let  $V$  be a vector space. The corresponding trivial vector bundle has the form  $M \times V$ . It is known, that the set of sections of  $M \times V$  is the set of all  $V$ -valued functions denoted by  $C(M) \otimes V$ . This classical background leads us to the following definition of a locally trivial vector bundle.

**Definition 6** *A locally trivial quantum vector bundle (QVB) is a tuple*

$$\{(E, B, \kappa), V, (\zeta_i, J_i)_{i \in I}\} \quad (3)$$

where  $(E, B, \kappa)$  is a faithful left module,  $(J_i)_{i \in I}$  is a complete covering of  $B$ ,  $V$  is a vector space and  $\zeta_i : E \longrightarrow B_i \otimes V$  are surjective left module homomorphisms with the properties

$$(\ker \zeta_i)_{i \in I} \quad \text{complete covering of } E \quad (4)$$

$$\zeta_i(\kappa(a)(e)) = \pi_i(a)\zeta_i(e) \quad a \in B, e \in E \quad (5)$$

$$\ker \zeta_i + \ker \zeta_j = \ker((\pi_j^i \otimes id) \circ \zeta_i) = \ker((\pi_i^j \otimes id) \circ \zeta_j). \quad (6)$$

Remark: In this definition we have used the left module structure  $(B_i \otimes V, B, \kappa_i)$ , which is defined by

$$\kappa_i(a)(b_i \otimes v) = \pi_i(a)b_i \otimes v \quad a \in B, b_i \in B_i, v \in V.$$

In the following we want to denote such a vector bundle by  $E$ .

By definition, for a locally trivial QVB  $E$  the family of submodules  $(\ker \zeta_i)_{i \in I}$  is a complete covering of  $E$ , i.e.  $E$  is isomorphic to its covering completion. Note that there are isomorphisms  $\tilde{\zeta}_i : E/\ker \zeta_i \longrightarrow B_i \otimes V$  defined by

$$\tilde{\zeta}_i(e + \ker \zeta_i) = \zeta_i(e).$$

Because of (6) there exist also isomorphisms  $\zeta_{ij}^i : E/(\ker \zeta_i + \ker \zeta_j) \longrightarrow B_{ij} \otimes V$  defined by

$$\zeta_{ij}^i(e + \ker \zeta_i + \ker \zeta_j) = \zeta_i(e) + \zeta_i(\ker \zeta_j),$$

thus there are left  $B_{ij}$ -module isomorphisms  $\phi_{ij_E}$  defined by  $\phi_{ij_E} := \zeta_{ij}^i \circ \zeta_{ij}^j^{-1}$  such that the covering completion of  $E$  has the form

$$E = \{(e_i)_{i \in I} \in \bigoplus_{i \in I} B_i \otimes V \mid (\pi_j^i \otimes id)(e_i) = \phi_{ij_E} \circ (\pi_i^j \otimes id)(e_j)\}.$$

**Proposition 3** *Let  $N$  be a locally trivial topological vector bundle over a compact topological space  $M$ .*

*Then the set of continuous sections  $\Gamma(N) = \{s : M \longrightarrow N\}$  is a locally trivial QVB.*

*Let  $B = C(M)$  be the algebra of continuous functions over a compact topological space  $M$ . Let  $(J_i)_{i \in I}$  be a finite covering of  $B$  coming from a finite covering  $(U_i)_{i \in I}$  of  $M$  by closed sets with nonempty open interior. Let  $E$  be a locally trivial QVB over  $B$  corresponding to this covering. Then  $E$  is the space of sections of a locally trivial vector bundle  $N$ .*

Proof: We want to give here only the idea of the proof. To prove the first assertion one defines the module homomorphisms  $\zeta_i$  in terms of the trivializations  $\psi_i : N \longrightarrow U_i \times V$  by

$$\zeta_i(s) = \psi_i \circ s$$

(identifying  $\psi_i(s(x)) = (x, v)$  with  $v$ ) and shows the conditions claimed for  $\zeta_i$ .

To prove the second assertion one constructs in terms of the given locally trivial QVB a locally trivial vector bundle in the following way. Let  $X = \bigcup_i \{i\} \times U_i \times V$ . One obtains a locally trivial vector bundle  $N$  over  $M$  by factorizing  $X$  with respect to the following relation:

$$(i, x, v) \sim (j, x', v') \text{ if } x = x' \text{ and } v = \phi_{ij_E}(1 \otimes v')(x).$$

One proves that the module of sections  $\Gamma(N)$  is isomorphic to  $E$ .  $\square$ .

Assume that there is given an LC differential algebra  $\Gamma(B)$  with complete covering  $(J_{i_\Gamma})_{i \in I}$  such that  $pr_0(J_{i_\Gamma}) = J_i$ , i.e. the factor algebras  $\Gamma(B)/J_{i_\Gamma} = \Gamma(B_i)$  are differential calculi over the factor algebras  $B_i = B/J_i$ . One can construct a locally trivial QVB  $((E_\Gamma, \Gamma(B), \kappa_\Gamma), V, (\zeta_{i_\Gamma}, J_{i_\Gamma})_{i \in I})$  from  $E$  in the following way. One extends  $\phi_{ij_E}$  to  $\Gamma(B_{ij}) \otimes V$  by

$$\phi_{ij_E}(\gamma \otimes v) = \gamma \phi_{ij_E}(1 \otimes v), \quad \gamma \in \Gamma(B_{ij}), \quad v \in V.$$

In terms of this extended module isomorphism one defines  $E_\Gamma$  by

$$E_\Gamma := \{(e_{i_\Gamma})_{i \in I} \in \bigoplus_{i \in I} \Gamma(B_i) \otimes V \mid (\pi_{j_\Gamma}^i \otimes id)(e_{i_\Gamma}) = \phi_{ij_E} \circ (\pi_{i_\Gamma}^j \otimes id)(e_{i_\Gamma})\}, \quad (7)$$

where the homomorphisms  $\pi_{j_\Gamma}^i : \Gamma(B_i) \longrightarrow \Gamma(B_{ij}) := \Gamma(B)/(J_{i_\Gamma} + J_{j_\Gamma})$  are the canonical projections.

By defining  $\kappa_\Gamma : \Gamma(B) \longrightarrow \text{End}(E_\Gamma)$  as

$$\kappa_\Gamma(\gamma)((e_{i_\Gamma})_{i \in I}) := (\pi_{i_\Gamma}(\gamma)e_{i_\Gamma})_{i \in I}$$

( $\pi_{i_\Gamma} : \Gamma(B) \longrightarrow \Gamma(B_i) = \Gamma(B)/J_{i_\Gamma}$  is the canonical projection.) and  $\zeta_{i_\Gamma} : E_\Gamma \longrightarrow \Gamma(B_i) \otimes V$  as the  $i$ -th projection one obtains a locally trivial QVB  $((E_\Gamma, \Gamma(B), \kappa_\Gamma), V, (\zeta_{i_\Gamma}, J_{i_\Gamma})_{i \in I})$ .

Now one can consider connections on such locally trivial QVB. We add to the usual definition of a connection in a “vector bundle” as a covariant derivative ([5]) a condition of compatibility with the covering.

**Definition 7** Let  $(E_\Gamma, \Gamma(B), \kappa_\Gamma), V, (\zeta_{i_\Gamma}, J_{i_\Gamma})_{i \in I}$  be the locally trivial QVB just defined. A connection on  $E_\Gamma$  is a linear map  $\nabla : E_\Gamma \longrightarrow E_\Gamma$  satisfying

$$\nabla(\gamma e) = (d\gamma)e + (-1)^n \gamma \nabla(e), \quad \gamma \in \Gamma^n(B), \quad e \in E_\Gamma \quad (8)$$

$$\nabla(\ker \zeta_{i_\Gamma}) \subset \ker \zeta_{i_\Gamma}, \quad \forall i \in I. \quad (9)$$

It is easy to see that from the property (9) follows that there exist connections  $\nabla_i$  on  $\Gamma(B_i) \otimes V$  and  $\nabla_{ij}^i$  on  $\Gamma(B_{ij}) \otimes V$  such that

$$\begin{aligned} \nabla_i \circ \zeta_{i_\Gamma} &= \zeta_{i_\Gamma} \circ \nabla \\ \nabla_{ij}^i \circ (\pi_{j_\Gamma}^i \otimes id) &= (\pi_{j_\Gamma}^i \otimes id) \circ \nabla_i. \end{aligned}$$

**Proposition 4** Connections on  $E_\Gamma$  are in one to one correspondence with families of connections  $\nabla_i : \Gamma(B_i) \otimes V \longrightarrow \Gamma(B_i) \otimes V$  satisfying

$$\nabla_{ij}^i = \phi_{ij_E} \circ \nabla_{ij}^j \circ \phi_{ji_E}. \quad (10)$$

Proof: Let  $\nabla$  be a connection on  $E_\Gamma$ . There exists a family of connections  $\nabla_i$  on  $\Gamma(B_i) \otimes V$  such that

$$\nabla((e_{i_\Gamma})_{i \in I}) = (\nabla_i(e_{i_\Gamma}))_{i \in I}.$$

Because of the identities

$$\begin{aligned} (\pi_{j_\Gamma}^i \otimes id)(e_{i_\Gamma}) &= \phi_{ij_E} \circ (\pi_{i_\Gamma}^j \otimes id)(e_{j_\Gamma}) \\ (\pi_{j_\Gamma}^i \otimes id) \circ \nabla_i(e_{i_\Gamma}) &= \phi_{ij_E} \circ (\pi_{i_\Gamma}^j \otimes id) \circ \nabla_j(e_{j_\Gamma}) \end{aligned}$$

one obtains

$$\begin{aligned} \nabla_{ij}^i \circ (\pi_{j_\Gamma}^i \otimes id)(e_{i_\Gamma}) &= \phi_{ij_E} \circ \nabla_{ij}^j \circ (\pi_{i_\Gamma}^j \otimes id)(e_{j_\Gamma}) \\ &= \phi_{ij_E} \circ \nabla_{ij}^j \circ \phi_{ji_E} \circ (\pi_{j_\Gamma}^i \otimes id)(e_{i_\Gamma}). \end{aligned}$$

Conversely, if there is given a family of connections  $\nabla_i$  satisfying property (10) the image of the linear map  $\nabla$  defined by

$$\nabla((e_{i_\Gamma})_{i \in I}) = (\nabla_i(e_{i_\Gamma}))_{i \in I}$$

lies in  $E_\Gamma$  and has the properties of a connection.  $\square$

Remark: Let the family  $(e_i)_{i \in I}$  be a linear basis in  $V$ . Let  $\nabla$  be a connection on  $\Gamma(B) \otimes H$ . Then there is a family of one forms  $(A_i^j)$  such that  $\nabla(1 \otimes e_i) = \sum_j A_i^j \otimes e_j$ .

**Definition 8** Let  $\nabla$  be a connection on  $E_\Gamma$ . We call the linear map  $\nabla^2$  the curvature of  $\nabla$ .

Note that  $\nabla^2(\gamma e) = \gamma \nabla^2(e)$ .

In the sequel we will be interested in QVB associated to a QPFB. We define these as follows (see also [2]):

**Definition 9** Let  $\mathcal{P}$  be a locally trivial QPFB, let  $F$  be a vector space and let  $\rho : F \longrightarrow H \otimes F$  be a left  $H$  coaction. The vector bundle  $E(\mathcal{P}, F)$  associated to  $\mathcal{P}$  and  $\rho$  is defined as

$$E(\mathcal{P}, F) := \{e \in \mathcal{P} \otimes F \mid (\Delta_{\mathcal{P}} \otimes id)(e) = (id \otimes \rho)(e)\}. \quad (11)$$

Remark: This is also called co-tensor product of  $\mathcal{P}$  and  $F$ .

**Proposition 5** The associated vector bundle  $E(\mathcal{P}, F)$  is a locally trivial QVB.

Proof: By formula (10) of [4].  $E(\mathcal{P}, F)$  has the form

$$\begin{aligned} E(\mathcal{P}, F) &= \{(e_i)_{i \in I} \in \bigoplus_{i \in I} B_i \otimes H \otimes F \mid (\pi_j^i \otimes id \otimes id)(e_i) = (\phi_{ij} \otimes id) \circ (\pi_i^j \otimes id \otimes id)(e_j); \\ &\quad (id \otimes \Delta \otimes id)(e_i) = (id \otimes id \otimes \rho)(e_i)\}. \end{aligned}$$

There are isomorphisms

$$id \otimes \varepsilon \otimes id : E_i := \{e_i \in B_i \otimes H \otimes F \mid id \otimes \Delta \otimes id(e_i) = id \otimes id \otimes \rho(e_i)\} \longrightarrow B_i \otimes F,$$

where  $(id \otimes \varepsilon \otimes id)^{-1} = id \otimes \rho$ , such that all  $e_i \in E_i$  are of the form

$$\sum_k \sum a_k \otimes f_{k(-1)} \otimes f_{k(0)}.$$

One easily verifies that  $((\tilde{E}, B, \tilde{\kappa}), F, (\tilde{\zeta}_i, J_i)_{i \in I})$  defined by

$$\tilde{E}(\mathcal{P}, F) := \{(\sum_k a_{k_i}^i \otimes f_{k_i}^i) \in \bigoplus_{i \in I} B_i \otimes F \mid \sum_k \pi_j^i(a_{k_i}^i) \otimes f_{k_i}^i = \sum_k \sum \pi_i^j(a_{k_i}^j) \tau_{ji}(f_{k_j(-1)}^j) \otimes f_{k_j(0)}^j\}$$

$$\begin{array}{ccc} \kappa(a)((\tilde{e}_i)_{i \in I}) & = & (\pi_i(a)\tilde{e}_i)_{i \in I}, \quad a \in B, \quad (\tilde{e}_i)_{i \in I} \in \tilde{E} \\ \zeta_i & & \text{i-th projection} \end{array}$$

is a locally trivial QVB (For the definition of the transition functions  $\tau_{ij}$  see [4]). In terms of the isomorphisms  $id \otimes \varepsilon \otimes id$  one can define a module isomorphism  $\epsilon : E(\mathcal{P}, F) \longrightarrow \tilde{E}(\mathcal{P}, F)$  by

$$\epsilon((e_i)_{i \in I}) := (id \otimes \varepsilon \otimes id(e_i))_{i \in I}.$$

This isomorphism exists due to the glueing properties. This is easy to see: An element  $(e_i)_{i \in I} \in E(\mathcal{P}, F)$  is of the form  $(e_i)_{i \in I} = (\sum \sum_{k_i} a_{k_i}^i \otimes f_{k_i(-1)}^i \otimes f_{k_i(0)}^i)_{i \in I}$  satisfying

$$\sum_{k_i} \sum_{k_j} \pi_j^i(a_{k_i}^i) \otimes f_{k_i(-1)}^i \otimes f_{k_i(0)}^i = \sum_{k_j} \sum_{k_j} \pi_i^j(a_{k_j}^j) \tau_{ji}(f_{k_j(-2)}^j) \otimes f_{k_j(-1)}^j \otimes f_{k_j(0)}^j.$$

(see formula (10) of [4])

Applying  $id \otimes \varepsilon \otimes id$  to both sides of this equation, one obtains for the element  $(id \otimes \varepsilon \otimes id(e_i))_{i \in I}$  the property

$$\sum_{k_i} \pi_j^i(a_{k_i}^i) \otimes f_{k_i}^i = \sum_{k_j} \sum_{k_j} \pi_i^j(a_{k_j}^j) \tau_{ji}(f_{k_j(-1)}^j) \otimes f_{k_j(0)}^j,$$

i.e. the image of  $\epsilon$  lies in  $\tilde{E}$ . The inverse of  $\epsilon$  is the map  $(\tilde{e}_i)_{i \in I} \longrightarrow (id \otimes \rho(\tilde{e}_i))_{i \in I}$ , i.e  $\epsilon$  is an isomorphism. We obtain the locally trivial QVB  $((E(\mathcal{P}, F), B, \kappa), F, (\zeta_i, J_i)_{i \in I})$  with

$$\begin{array}{ccc} \kappa & = & \tilde{\kappa} \circ \epsilon \\ \zeta_i & = & \tilde{\zeta}_i \circ \epsilon. \end{array}$$

□

As in the general case of locally trivial vector bundles one can construct a locally trivial vector bundle  $((E_\Gamma(\mathcal{P}, F), \Gamma_m(B), \kappa_\Gamma), (\zeta_{i_\Gamma}, \ker \pi_{i_\Gamma m})_{i \in I})$  from the associated vector bundle  $E(\mathcal{P}, F)$ . The LC differential algebra  $\Gamma_m(B)$  is the maximal embeddable LC-differential algebra induced from the differential structure  $\Gamma_c(\mathcal{P})$  on the locally trivial QPFB  $\mathcal{P}$  (see [3] and [4]). Let  $\phi_{ij_E} : B_{ij} \otimes F \longrightarrow B_{ij} \otimes F$  be defined by

$$\phi_{ij_E}(a \otimes f) := a \tau_{ji}(f_{(-1)}) \otimes f_{(0)}.$$

By definition,

$$E_\Gamma(\mathcal{P}, F) = ((e_{i_\Gamma})_{i \in I} \in \bigoplus_{i \in I} \Gamma(B_i) \otimes F | (\pi_{j_\Gamma m}^i \otimes id)(e_{i_\Gamma}) = \phi_{ij_E} \circ (\pi_{i_\Gamma m}^j \otimes id)(e_{j_\Gamma})). \quad (12)$$

**Proposition 6** *Let*

$$horE(\mathcal{P}, F) := \{\gamma_E \in hor\Gamma_c(\mathcal{P}) \otimes F | (\Delta_{\mathcal{P}_\Gamma} \otimes id)(\gamma_E) = (id \otimes \rho)(\gamma_E)\}.$$

*horE(\mathcal{P}, F) and  $E_\Gamma(\mathcal{P}, F)$  are isomorphic as left  $\Gamma_m(B)$ -modules.*

Remark:  $horE(\mathcal{P}, F)$  is in the classical situation the space of horizontal forms “of type  $\rho$ ”.

Proof: By definition of  $hor\Gamma_c(\mathcal{P})$ ,

$$\begin{aligned} horE(\mathcal{P}, F) &= \{(\gamma_{i_E})_{i \in I} \in \bigoplus_{i \in I} \Gamma(B_i) \hat{\otimes} H \otimes F | \\ &((\pi_j^i \otimes id)_\Gamma \otimes id)(\gamma_{i_E}) = (\phi_{ij_\Gamma} \otimes id) \circ ((\pi_i^j \otimes id)_\Gamma \otimes id)(\gamma_{j_E}); \\ &(id \otimes \Delta \otimes id)(\gamma_{i_E}) = (id \otimes id \otimes \rho)(\gamma_{i_E})\}. \end{aligned}$$

The last condition means that the  $i$ -th components have the form

$$\gamma_{i_E} = \sum_{k_i} \sum \gamma_{k_i}^i \otimes f_{k_i(-1)}^i \otimes f_{k_i(0)}^i.$$

Now one defines again an isomorphism  $\epsilon_\Gamma : \text{hor}E(\mathcal{P}, F) \longrightarrow E_\Gamma(\mathcal{P}, F)$  by

$$\epsilon_\Gamma((\gamma_{i_E})_{i \in I}) = (id \otimes \varepsilon \otimes id(\gamma_{i_E}))_{i \in I}.$$

This isomorphism exists, if one can show that from the gluing conditions in  $\text{hor}E(\mathcal{P}, F)$  the gluing conditions in  $E_\Gamma(\mathcal{P}, F)$  follow. To this end apply  $P_{inv} \otimes id$  (formula (66) of [4]) to

$$\sum_{k_i} \sum (\pi_j^i \otimes id)_\Gamma (\gamma_{k_i}^i \otimes f_{k_i(-1)}^i) \otimes f_{k_i(0)}^i = \sum_{k_j} \sum \phi_{ij_\Gamma} \circ (\pi_i^j \otimes id)_\Gamma (\gamma_{k_j}^j \otimes f_{k_j(-1)}^j) \otimes f_{k_j(0)}^j.$$

By the definition  $\pi_{j_\Gamma}^i(\gamma) = (\pi_j^i \otimes id)_\Gamma(\gamma \hat{\otimes} 1)$  one obtains the gluing condition in  $E_\Gamma(\mathcal{P}, F)$ , i.e.

$$\sum_{k_i} \pi_{j_\Gamma}^i(\gamma_{k_i}^i) \otimes f_{k_i}^i = \sum_{k_j} \phi_{ij_{E_\Gamma}}(\pi_{i_\Gamma}^j(\gamma_{k_j}^j) \otimes f_{k_j}^j),$$

which means that the image of  $\epsilon_\Gamma$  lies in  $E_\Gamma(\mathcal{P}, F)$ . Conversely, the inverse of  $\epsilon_\Gamma$  is obviously defined by

$$\epsilon_\Gamma^{-1}((e_{i_\Gamma})_{i \in I}) = (id \otimes \rho(e_{i_\Gamma}))_{i \in I}.$$

□

Now we are interested in connections on such associated vector bundles. An important class of connections are the connections induced from left left covariant derivations on the locally trivial QPFB.

**Proposition 7** *Every left covariant derivation on the locally trivial QPFB  $\mathcal{P}$  determines uniquely a connection on  $E_\Gamma(\mathcal{P}, F)$ .*

Proof: One defines a linear map  $\nabla : E_\Gamma \longrightarrow E_\Gamma$  by

$$\nabla := \epsilon_\Gamma \circ (D_l \otimes id) \circ \epsilon_\Gamma^{-1},$$

which is easily seen to be a connection on  $E_\Gamma(\mathcal{P}, F)$ . □

Remark: Because of the bijection between left and right covariant derivations also right covariant derivations on  $\mathcal{P}$  determine connections on  $E_\Gamma(\mathcal{P}, F)$ .

The curvature of such a connection is related to the curvature on the locally trivial QPFB  $\mathcal{P}$  by

$$\nabla^2 = \epsilon_\Gamma \circ (D_l^2 \otimes id) \circ \epsilon_\Gamma^{-1}.$$

The corresponding connections  $\nabla_i : \Gamma(B_i) \otimes F \longrightarrow \Gamma(B_i) \otimes F$  have the form

$$\nabla_i(\gamma \otimes f) = d\gamma \otimes f - \sum (-1)^n \gamma A_{l_i}(f_{-1}) \otimes f_0, \quad \gamma \in \Gamma^n(B_i), \quad f \in F.$$

The curvatures of these connections are

$$\nabla_i^2(\gamma \otimes f) = \sum \gamma F_{l_i}(f_{-1}) \otimes f_0.$$



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